Lp-Theory for Vector Potentials and Sobolev's Inequalities for Vector Fields

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Abstract

In a three dimensional bounded possibly multiply-connected domain, we prove the existence and uniqueness of vector potentials in L^p theory, associated with a divergence-free function and satisfying some boundary conditions. We also present some results concerning scalar potentials and weak vector potentials. Furthermore, various Sobolev-type inequalities are given.

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Résumé

Théorie L^p pour les potentiels vecteurs et inégalités de Sobolev pour des champs de vecteurs. Dans un ouvert borné tridimensionnel, éventuellement multiplement connexe, nous prouvons l'existence et l'unicité des potentiels vecteurs en théorie L^p, associés à des champs de vecteurs à divergence nulle et vérifiant plusieurs conditions aux limites. On présente également des résultats concernant les potentiels scalaires et les potentiels vecteurs faibles. De plus, plusieurs inégalités de Sobolev sont données. *Pour citer cet article : C. Amrouche, N. Seloula, C. R. Acad. Sci. Paris, Ser. I 340 (2010).*

Version française abrégée

Dans cette Note on s'intéresse à la théorie des potentiels vecteurs dans un ouvert Ω borné tridimensionnel éventuellement non simplement connexe à bord Γ de classe $\mathcal{C}^{1,1}$. Le cadre hilbertien est déjà traité par C. Amrouche, C. Bernardi, M. Dauge et V. Girault [1]. L'originalité de notre travail est de développer

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des résultats similaires en théorie L^p lorsque $1 < p < \infty$. Le résultat de base concernant l'existence d'un potentiel vecteur sans conditions aux limites est donné dans le Théorème 3.1. En particulier dans le Théorème 3.2, le Théorème 3.3 et le Théorème 3.4, plusieurs conditions aux limites sont proposées. Les autres résultats concernent la régularité de tels potentiels vecteurs. On s'intéresse ensuite au cas des potentiels scalaires et potentiels vecteurs très faibles.

1. Introduction

In this work, we assume that Ω is a bounded open connected set of $\mathbb{R}^3$ of class $C^{1,1}$ with boundary Γ. Let Γ_i, $0 \le i \le I$, denote the connected components of the boundary Γ, Γ_0 being the exterior boundary of Ω. We do not assume that Ω is simply-connected but we suppose that there exist J connected open surfaces Σ_j, $1 \le j \le J$, called 'cuts', contained in Ω, such that each surface Σ_j is an open subset of a smooth manifold. The boundary of each Σ_j is contained in Γ. The intersection $\overline{\Sigma_i} \cap \overline{\Sigma_j}$ is empty for $i \neq j$, and finally the open set $\Omega^\circ = \Omega \setminus \cup_{j=1}^{J} \Sigma_j$ is simply-connected. We denote by $[\cdot]_j$ the jump of a function over Σ_j, for $1 \le j \le J$. The pair $\langle \cdot, \cdot \rangle_{X,X'}$ denotes the duality product between a space X and X'. We then define the spaces:

$$\boldsymbol{H}^{p}(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\, \mathbf{curl}\ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\}\,,\ \ \boldsymbol{H}^{p}(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\, \mathrm{div}\ \boldsymbol{v} \in L^p(\Omega)\}$$

$$\boldsymbol{X}^{p}(\Omega) = \boldsymbol{H}^{p}(\mathbf{curl}, \Omega) \cap \boldsymbol{H}^{p}(\mathrm{div}, \Omega),$$

equipped with the graph norm. We also define their subspaces:

$$\boldsymbol{H}_0^{p}(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^{p}(\mathbf{curl}, \Omega);\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\}\,,$$

$$\boldsymbol{H}_0^{p}(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^{p}(\mathrm{div}, \Omega);\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}\,,$$

$$\boldsymbol{X}_N^{p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}^{p}(\Omega);\, \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\}\,,\quad \boldsymbol{X}_T^{p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}^{p}(\Omega);\, \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}$$

and $\boldsymbol{X}_0^{p}(\Omega) = \boldsymbol{X}_N^{p}(\Omega) \cap \boldsymbol{X}_T^{p}(\Omega)$. We also define the space

$$\boldsymbol{W}_{\sigma}^{1,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega),\ \mathrm{div}\, \boldsymbol{v} = 0 \text{ in}\, \Omega\}.$$

As in the Hilbertian case, we can prove that the space $\boldsymbol{X}_0^{p}(\Omega)$ coincides with $\boldsymbol{W}_0^{1,p}(\Omega)$ for $1 < p < \infty$. We can also prove that $\mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{H}^{p}(\mathbf{curl}, \Omega)$, $\boldsymbol{H}^{p}(\mathrm{div}, \Omega)$ and $\boldsymbol{X}^{p}(\Omega)$. Also $\mathcal{D}(\Omega)$ is dense in $\boldsymbol{H}_0^{p}(\mathbf{curl}, \Omega)$ and $\boldsymbol{H}_0^{p}(\mathrm{div}, \Omega)$. For any function q in $W^{1,p}(\Omega^\circ)$, $\mathbf{grad}\, q$ can be extended to $\boldsymbol{L}^p(\Omega)$. We denote this extension by $\widetilde{\mathbf{grad}}\, q$. We finally define the spaces:

$$\boldsymbol{K}_T^{p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}_T^{p}(\Omega),\ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0},\ \mathrm{div}\, \boldsymbol{v} = 0 \text{ in}\, \Omega\}\,,$$

$$\boldsymbol{K}_N^{p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}_N^{p}(\Omega),\ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0},\ \mathrm{div}\, \boldsymbol{v} = 0 \text{ in}\, \Omega\}\,.$$

As shown in [1], Proposition 3.14, for the case $p = 2$, we can prove that the space $\boldsymbol{K}_T^{p}(\Omega)$ is of dimension J and is spanned by the functions $\widetilde{\mathbf{grad}}\, q_j^T$, $1 \le j \le J$, where each $q_j^T \in W^{1,p}(\Omega^\circ)$ is unique up to an additive constant and satisfies $\Delta\, q_j^T = 0$ in Ω°, $\partial_n\, q_j^T = 0$ on Γ, $\left[\, q_j^T \,\right]_k =$ constant, $[\, \partial_n\, q_j^T \,]_k = 0$; $1 \le k \le J$ and $\left\langle \partial_n\, q_j^T,\, 1 \right\rangle_{\Sigma_k} = \delta_{j\,k}$, $1 \le k \le J$. We note that $\boldsymbol{K}_T^{p}(\Omega) = \{0\}$ if $J = 0$, where J is the second Betti number. Similarly, we can prove that the dimension of the space $\boldsymbol{K}_N^{p}(\Omega)$ is I and that it is spanned by the functions $\mathbf{grad}\, q_i^N$, $1 \le i \le I$, where each $q_i^N \in W^{1,p}(\Omega)$ is the unique solution to the problem $\Delta\, q_i^N = 0$ in Ω, $q_i^N = 0$ in Γ_0, $q_i^N =$ constant in Γ_k, $\langle \partial_n\, q_i^N,\, 1 \rangle_{\Gamma_0} = -1$ and $\langle \partial_n\, q_i^N,\, 1 \rangle_{\Gamma_k} = \delta_{i\,k}$, $1 \le k \le I$. We note that I is the first Betti number and if $\Gamma = \Gamma_0$, then $\boldsymbol{K}_N^{p}(\Omega) = \{0\}$. In the sequel, the letter C denotes a constant that is not necessarily the same at its various occurences. The detailled proofs of the results announced in this Note are given in [3].

2. Sobolev's inequality and compactness properties

We introduce the following two operators:

$$T\,\lambda(\boldsymbol{x}) = -\frac{1}{2\pi}\int_\Gamma \lambda(\boldsymbol{\xi})\,\frac{\partial}{\partial \boldsymbol{n}}|\boldsymbol{x}-\boldsymbol{\xi}|^{-1}\,\mathrm{d}\sigma_{\boldsymbol{\xi}}, \qquad R\,\boldsymbol{\lambda}(\boldsymbol{x}) = \frac{1}{2\pi}\int_\Gamma \mathbf{curl}\,\big(\frac{\boldsymbol{\lambda}(\boldsymbol{\xi})}{|\boldsymbol{x}-\boldsymbol{\xi}|}\big)\times\boldsymbol{n}\,\mathrm{d}\sigma_{\boldsymbol{\xi}},$$

where T is compact from $L^p(\Gamma)$ into $L^p(\Gamma)$ and R is compact from $\boldsymbol{L}^p(\Gamma)$ into $\boldsymbol{L}^p(\Gamma)$ (see [6]). Using the Fredholm alternative, we can check that the null spaces $\mathrm{Ker}(Id+T)$ and $\mathrm{Ker}(Id+R)$ are of finite dimension and are respectively spanned by the traces of the functions $\mathbf{grad}\,q_i^N\cdot\boldsymbol{n}$ on Γ for any $1\le i\le I$ and the traces of the functions $\widetilde{\mathbf{grad}}\,q_j^T\times\boldsymbol{n}$ on Γ for any $1\le j\le J$. The next lemma is a generalization of the one in [6] to the case $I\ge 1$ and $J\ge 1$. We expect that to estimate $\nabla\,\boldsymbol{v}$, in addition to $\mathrm{div}\,\boldsymbol{v}$ and $\mathbf{curl}\,\boldsymbol{v}$, the quantity $\sum_{j=1}^{J}\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}$ is necessarily in the case where $\boldsymbol{v}\cdot\boldsymbol{n}$ vanish on Γ (respectively the quantity $\sum_{i=1}^{I}\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}$ is necessarily in the case where $\boldsymbol{v}\times\boldsymbol{n}$ vanish on Γ).

Lemma 2.1.

i) Any function $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)\cap\boldsymbol{X}_N^p(\Omega)$ *satisfies:*

$$\|\nabla\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\le C\Big(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)}+\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\sum_{i=1}^{I}|\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}|\Big).$$

ii) Any function $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)\cap\boldsymbol{X}_T^p(\Omega)$ *satisfies:*

$$\|\nabla\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\le C\Big(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)}+\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\sum_{j=1}^{J}|\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}|\Big).$$

Sketch of the proof. We use the integral representations, the properties of the operators T and R and the Calderon-Zygmund inequalities. □

Using the previous result, the density of $\boldsymbol{W}^{1,p}(\Omega)\cap\boldsymbol{X}_N^p(\Omega)$ in $\boldsymbol{X}_N^p(\Omega)$ and the density of $\boldsymbol{W}^{1,p}(\Omega)\cap\boldsymbol{X}_T^p(\Omega)$ in $\boldsymbol{X}_T^p(\Omega)$, we obtain the following theorem:

Theorem 2.2. *The spaces* $\boldsymbol{X}_N^p(\Omega)$ *and* $\boldsymbol{X}_T^p(\Omega)$ *are both continuously imbedded in* $\boldsymbol{W}^{1,p}(\Omega)$ *and we have the following estimates:*

i) Any $\boldsymbol{v}\in\boldsymbol{X}_N^p(\Omega)$ *satisfies:*

$$\|\,\boldsymbol{v}\,\|_{\boldsymbol{W}^{1,p}(\Omega)}\le C\Big(\|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^p(\Omega)}+\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)}+\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\sum_{i=1}^{I}|\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}|\Big).$$

ii) Any $\boldsymbol{v}\in\boldsymbol{X}_T^p(\Omega)$ *satisfies:*

$$\|\,\boldsymbol{v}\,\|_{\boldsymbol{W}^{1,p}(\Omega)}\le C\Big(\|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^p(\Omega)}+\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)}+\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\sum_{j=1}^{J}|\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}|\Big).$$

We now introduce the following spaces for $s\in\mathbb{R}$, $s\ge 1$:

$$\boldsymbol{X}^{s,p}(\Omega)=\{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega);\ \mathrm{div}\,\boldsymbol{v}\in W^{s-1,p}(\Omega),\ \mathbf{curl}\,\boldsymbol{v}\in\boldsymbol{W}^{s-1,p}(\Omega)\ \text{and}\ \boldsymbol{v}\cdot\boldsymbol{n}\in\boldsymbol{W}^{s-\frac{1}{p},p}(\Gamma)\},$$

$$\boldsymbol{Y}^{s,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div} \boldsymbol{v} \in W^{s-1,p}(\Omega),\ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{W}^{s-1,p}(\Omega) \text{ and } \boldsymbol{v} \times \boldsymbol{n} \in \boldsymbol{W}^{s-\frac{1}{p},p}(\Omega)\}.$$

The following result is an extension of Theorem 2.2 to the case where the boundary conditions $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ and $\boldsymbol{v} \times \boldsymbol{n} = 0$ on Γ are replaced by inhomogeneous ones.

Theorem 2.3. *The spaces $\boldsymbol{X}^{1,p}(\Omega)$ and $\boldsymbol{Y}^{1,p}(\Omega)$ are both continuously imbedded in $\boldsymbol{W}^{1,p}(\Omega)$:*

i) Any $\boldsymbol{v}$ in $\boldsymbol{X}^{1,p}(\Omega)$ satisfies

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{L^p(\Omega)} + \|\boldsymbol{v} \cdot \boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\big).$$

ii) Any $\boldsymbol{v}$ in $\boldsymbol{Y}^{1,p}(\Omega)$ satisfies

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\Big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{L^p(\Omega)} + \|\boldsymbol{v} \times \boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\Big).$$

iii) Let $m \in \mathbb{N}^*$ and Ω of class $\mathcal{C}^{m,1}$. Then, the space $\boldsymbol{X}^{m,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{m,p}(\Omega)$ and $\boldsymbol{Y}^{m,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{m,p}(\Omega)$).

iv) Let $s = m + \sigma$, $m \in \mathbb{N}^*$ and $0 < \sigma \leq 1$, Assume that Ω is of class $\mathcal{C}^{m+1,1}$. Then, the space $\boldsymbol{X}^{s,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{s,p}(\Omega)$ and $\boldsymbol{Y}^{s,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{s,p}(\Omega)$).

The first point of the following result is proven by Costabel [4] for the case $p = 2$ in a bounded simply connected domain. We extend this result to the case of a multiply connected domain Ω and for any $1 < p < \infty$.

Theorem 2.4.

(i) Let $\boldsymbol{u} \in \boldsymbol{X}^p(\Omega)$ with $\boldsymbol{u} \cdot \boldsymbol{n} \in L^p(\Gamma)$ (respectively with $\boldsymbol{u} \times \boldsymbol{n} \in L^p(\Gamma)$). Then, $\boldsymbol{u} \in \boldsymbol{W}^{\frac{1}{p},p}(\Omega)$ and satisfies the inequality

$$\|\,\boldsymbol{u}\,\|_{\boldsymbol{W}^{\frac{1}{p},p}(\Omega)} \leq C\big(\|\,\boldsymbol{u}\,\|_{\boldsymbol{L}^p(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{u}\,\|_{\boldsymbol{L}^p(\Omega)} + \|\,\operatorname{div}\boldsymbol{u}\,\|_{L^p(\Omega)} + \|\,\boldsymbol{u} \cdot \boldsymbol{n}\,\|_{L^p(\Gamma)}\big),$$

$$\big(\textit{respectively } \|\,\boldsymbol{u}\,\|_{\boldsymbol{W}^{\frac{1}{p},p}(\Omega)} \leq C\big(\|\,\boldsymbol{u}\,\|_{\boldsymbol{L}^p(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{u}\,\|_{\boldsymbol{L}^p(\Omega)} + \|\,\operatorname{div}\boldsymbol{u}\,\|_{L^p(\Omega)} + \|\,\boldsymbol{u} \times \boldsymbol{n}\,\|_{L^p(\Gamma)}\big)\big).$$

(ii) If in addition $\boldsymbol{u} \cdot \boldsymbol{n} \in W^{s-1/p,p}(\Gamma)$ (respectively $\boldsymbol{u} \times \boldsymbol{n} \in W^{s-1/p,p}(\Gamma)$) with $1/p < s \leq 1$, then $\boldsymbol{u} \in \boldsymbol{W}^{s,p}(\Omega)$.

As for the case $p = 2$ (see [1]), we can prove that the imbedding of $\boldsymbol{X}^p(\Omega)$ into $\boldsymbol{L}^p(\Omega)$ is not compact and that the homogeneous normal or tangential boundary conditions are sufficient to insure compactness. More precisely we have the following result.

Theorem 2.5. *The imbeddings of $\boldsymbol{X}^p_N(\Omega)$ and $\boldsymbol{X}^p_T(\Omega)$ into $\boldsymbol{L}^p(\Omega)$ are compact.*

3. Vector potentials

We begin with a first result concerning vector potentials without boundary conditions. The result is known for $p = 2$ (see [1]) and we can give a different proof for $1 < p < \infty$ by using the fundamental solution of the Laplacian.

Theorem 3.1. *A vector field $\boldsymbol{u}$ in $\boldsymbol{H}^p(\operatorname{div}, \Omega)$ satisfies*

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in}\, \Omega \quad \text{and} \qquad \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_{\mathrm{i}}} = 0, \quad 0 \leq \mathrm{i} \leq \mathrm{I}, \tag{1}$$

if and only if there exists a vector potential $\boldsymbol{\psi}_0$ in $\boldsymbol{W}^{1,p}(\Omega)$ such that

$$\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi}_0.$$

Moreover, we can choose $\boldsymbol{\psi_0}$ *such that* $\operatorname{div} \boldsymbol{\psi}_0 = 0$ *and we have the estimate*

$$\|\boldsymbol{\psi}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\, \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)},$$

where $C > 0$ *depends only on* p *and* Ω.

Theorem 3.2. *A function* $\boldsymbol{u}$ *in* $\boldsymbol{H}^p(\operatorname{div}, \Omega)$ *satisfies* (1) *if and only if there exists a vector potential* $\boldsymbol{\psi}$ *in* $\boldsymbol{W}^{1,p}(\Omega)$ *such that*

$$\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi} \quad \text{and} \quad \operatorname{div} \boldsymbol{\psi} = 0 \text{ in } \Omega, \quad \boldsymbol{\psi} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma, \quad \langle \boldsymbol{\psi} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_{\mathrm{j}}} = 0,\ 1 \leq \mathrm{j} \leq \mathrm{J}. \tag{2}$$

This function $\boldsymbol{\psi}$ *is unique and we have the estimate:*

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3}$$

Sketch of the proof. Let $\boldsymbol{\psi}_0 \in \boldsymbol{W}^{\,1,p}(\Omega)$ be the function associated with $\boldsymbol{u}$ by Theorem 3.1 and $\chi \in W^{\,1,p}(\Omega)$ be a unique solution up to an additive constant, of the problem: $-\Delta\, \chi = 0$ in Ω and $\partial_n\, \chi = \boldsymbol{\psi}_0 \cdot \boldsymbol{n}$ on Γ. Then, the vector function

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 - \mathbf{grad}\, \chi - \sum_{j=1}^{J} \langle\, (\boldsymbol{\psi}_0 - \mathbf{grad}\, \chi) \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j}\, \widetilde{\mathbf{grad}}\, q_j^T,$$

satisfies the properties in (2) and the estimate (3). □

Applying the Peetre-Tartar Lemma, (cf. references [5], Chapter I, Theorem 2.1), we can prove the following Poincaré-type inequality.

Corollary 3.3. *On the space* $\boldsymbol{X}_T^p(\Omega)$*, the seminorm*

$$\boldsymbol{w} \mapsto \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{w}\|_{L^p(\Omega)} + \sum_{j=1}^{J} |\langle \boldsymbol{w} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j}|,$$

is equivalent to the norm $\|\cdot\|_{\boldsymbol{X}^p(\Omega)}$.

Theorem 3.4. *A function* $\boldsymbol{u}$ *in* $\boldsymbol{H}^p(\operatorname{div}, \Omega)$ *satisfies:*

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in}\, \Omega,\ \ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma \quad \text{and} \quad \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_{\mathrm{j}}} = 0, \quad 1 \leq \mathrm{j} \leq \mathrm{J}, \tag{4}$$

if and only if there exists a vector potential $\boldsymbol{\psi}$ *in* $\boldsymbol{W}^{1,p}(\Omega)$ *such that*

$$\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi} \quad \text{and} \quad \operatorname{div} \boldsymbol{\psi} = 0 \text{ in } \Omega, \quad \boldsymbol{\psi} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma, \quad \langle \boldsymbol{\psi} \cdot \boldsymbol{n},\ 1\rangle_{\Gamma_{\mathrm{i}}} = 0,\ \text{for any}\ 0 \leq \mathrm{i} \leq \mathrm{I}. \tag{5}$$

This function $\boldsymbol{\psi}$ *is unique and we have the estimate:*

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{6}$$

Sketch of the proof. We solve the problem $-\Delta\, \boldsymbol{\xi} = 0$ and $\operatorname{div} \boldsymbol{\xi} = 0$ in Ω, $\boldsymbol{\xi} \cdot \boldsymbol{n} = 0$, $\mathbf{curl}\, \boldsymbol{\xi} \times \boldsymbol{n} = \boldsymbol{\psi}_0 \times \boldsymbol{n}$ on Γ, $\langle \boldsymbol{\xi} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0$, where $\boldsymbol{\psi}_0$ is the function associated with $\boldsymbol{u}$ by Theorem 3.1. This problem is equivalent to the following: find $\boldsymbol{\xi} \in \boldsymbol{V}_T^p(\Omega)$ such that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{V}_T^{p'}(\Omega), \int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{\psi}_0 \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{\varphi} \cdot \mathbf{curl}\, \boldsymbol{\psi}_0\, \mathrm{d}\boldsymbol{x}, \tag{7}$$

where $\boldsymbol{V}_T^p(\Omega) = \{\boldsymbol{w} \in \boldsymbol{X}_T^p(\Omega);\, \operatorname{div} \boldsymbol{w} = 0 \text{ in}\, \Omega \text{ and } \langle \boldsymbol{w} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0,\ 1 \leq j \leq J\}$. We can prove that (7) satisfies the following Inf-Sup condition:

$$\inf_{\boldsymbol{\varphi} \in \boldsymbol{V}_T^{p'}(\Omega)} \sup_{\boldsymbol{\xi} \in \boldsymbol{V}_T^p(\Omega)} \frac{\int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\xi}\|_{\boldsymbol{X}_T^p(\Omega)} \|\boldsymbol{\varphi}\|_{\boldsymbol{X}_T^{p'}(\Omega)}} > 0, \tag{8}$$

and that the solution $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$. Then, the vector function

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 - \mathbf{curl}\,\boldsymbol{\xi} - \sum_{i=1}^{I} \langle (\boldsymbol{\psi}_0 - \mathbf{curl}\,\boldsymbol{\xi}) \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i}\, \mathbf{grad}\, q_i^N,$$

satisfies (5) and the estimate (6). □

Again, using the Peetre-Tartar Lemma, we have the following Poincaré-type inequality.

Corollary 3.5. *On the space $\boldsymbol{X}^p_N(\Omega)$, the seminorm*

$$\boldsymbol{w} \mapsto \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{w}\|_{L^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{w} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i}|,$$

is equivalent to the norm $\|\cdot\|_{\boldsymbol{X}^p(\Omega)}$.

In a similar fashion and by characterizing the kernel space

$$\boldsymbol{K}^p_0(\Omega) = \{\boldsymbol{w} \in \boldsymbol{W}^{1,p}_0(\Omega);\ \mathbf{curl}\,\boldsymbol{w} = \mathbf{0} \text{ and } \mathrm{div}\,(\Delta \boldsymbol{w}) = 0 \text{ in}\, \Omega\},$$

we can give another type of vector potential:

Theorem 3.6. *A function $\boldsymbol{u}$ in $\boldsymbol{H}^p(\mathrm{div},\, \Omega)$ satisfies:*

$$\mathrm{div}\,\boldsymbol{u} = 0 \quad \text{in}\, \Omega,\ \ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on}\ \Gamma \quad \text{and} \quad \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_\mathrm{j}} = 0, \quad 1 \le \mathrm{j} \le \mathrm{J},$$

if and only if there exists a vector potential $\boldsymbol{\psi}$ in $\boldsymbol{W}^{1,p}(\Omega)$ such that

$$\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi} \quad \text{and} \quad \mathrm{div}\,(\Delta \boldsymbol{\psi}) = 0 \text{ in } \Omega, \quad \boldsymbol{\psi} = 0 \ \text{ on}\, \Gamma, \quad \langle \partial_\mathrm{n}(\mathrm{div}\,\boldsymbol{\psi}),\, 1 \rangle_{\Gamma_\mathrm{i}} = 0, \text{ for any } 0 \le \mathrm{i} \le \mathrm{I}.$$

This function $\boldsymbol{\psi}$ is unique.

4. Scalar potentials

In this section we present several results concerning scalar potentials. We begin with the following fundamental result.

Theorem 4.1. *For any function $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ that satisfies*

$$\mathbf{curl}\,\boldsymbol{f} = \mathbf{0} \quad \text{and} \quad \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} = 0 \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{K}^{p'}_T(\Omega),$$

there exists a unique scalar potential $\chi \in W^{1,p}(\Omega)/\mathbb{R}$ such that $\boldsymbol{f} = \mathbf{grad}\,\chi$ and the following estimate holds:

$$\|\chi\|_{W^{1,p}(\Omega)/\mathbb{R}} \le C \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

Sketch of the proof. It suffices to prove that for any $\boldsymbol{v} \in \boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)$ such that $\mathrm{div}\, \boldsymbol{v} = 0$ in Ω, we have $\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} = 0$. Next, we apply [2, lemma 2.7]. □

We are now interested in the case of singular data.

Theorem 4.2. *For any $\boldsymbol{f}$ in the dual space of $\boldsymbol{H}^{p'}_0(\mathrm{div}\,, \Omega)$ with $\mathbf{curl}\,\boldsymbol{f} = 0$ in Ω and $\boldsymbol{f}$ satisfying:*

$$\forall\, \boldsymbol{v} \in \boldsymbol{K}^{p'}_T(\Omega), \quad \langle \boldsymbol{f},\, \boldsymbol{v} \rangle_{[\boldsymbol{H}^{p'}_0(\mathrm{div}\,, \Omega)]' \times \boldsymbol{H}^{p'}_0(\mathrm{div}\,, \Omega)} = 0,$$

there exists a scalar potential χ in $L^p(\Omega)/\mathbb{R}$ such that $\boldsymbol{f} = \mathbf{grad}\,\chi$ and the following estimate holds:

$$\|\chi\|_{L^p(\Omega)/\mathbb{R}} \le C \|\boldsymbol{f}\|_{\boldsymbol{H}^{p'}_0(\mathrm{div}\,, \Omega)'}.$$

Sketch of the proof. We can prove that for any $\boldsymbol{f}$ in $[\boldsymbol{H}^{p'}_0(\mathrm{div}\,, \Omega)]'$, there exists $\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)$ and $\chi_0 \in L^p(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{grad}\, \chi_0$. We then apply Theorem 4.1 to $\boldsymbol{\psi}$. □

5. Weak vector potentials

The aim of this section is the proof of the existence of a new type of vector potential called weak vector potentials.

Theorem 5.1. *For any* $\boldsymbol{f}$ *in the dual space of* $\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)$ *with* $\operatorname{div}\boldsymbol{f}=0$ in Ω *and* $\boldsymbol{f}$ *satisfying:*

$$\forall \boldsymbol{v}\in \boldsymbol{K}_N^{p'}(\Omega),\quad \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)]'\times \boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)}=0,$$

there exists a vector potential $\boldsymbol{\xi}$ *in* $\boldsymbol{L}^p(\Omega)$ *such that*

$$\boldsymbol{f}=\mathbf{curl}\,\boldsymbol{\xi},\qquad \text{with}\qquad \operatorname{div}\boldsymbol{\xi}=0\quad \text{in}\,\Omega\quad\text{and}\quad \boldsymbol{\xi}\cdot\boldsymbol{n}=0\quad\text{on}\,\Gamma,$$

and such that the following estimate holds:

$$\|\boldsymbol{\xi}\|_{\boldsymbol{L}^p(\Omega)}\leq C\|\boldsymbol{f}\|_{\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)'}.$$

Sketch of the proof. We can prove that for any $\boldsymbol{f}$ in $[\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)]'$, there exists $\boldsymbol{\psi}\in\boldsymbol{L}^p(\Omega)$ and $\boldsymbol{\xi}_0\in\boldsymbol{L}^p(\Omega)$ such that $\boldsymbol{f}=\boldsymbol{\psi}+\mathbf{curl}\,\boldsymbol{\xi}_0$ with $\operatorname{div}\boldsymbol{\xi}_0=0$ in Ω and $\boldsymbol{\xi}_0\cdot\boldsymbol{n}=0$ on Γ. We then apply Theorem 3.2 to $\boldsymbol{\psi}$. □

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